Week 5
Integral Polyhedra

We have seen some examples\(^1\) of linear programming formulation that are integral, meaning that every basic feasible solution is an integral vector. This week we develop a theory of integral polyhedra that explains the integrality of these example by a simple property of the constraint matrix of these programs.

5.1 Totally Unimodular Matrices

Consider the integer program

\[
\begin{align*}
\text{minimize} & \quad c \cdot x \\
\text{subject to} & \quad Ax = b \\
& \quad x \in \mathbb{Z}^n
\end{align*}
\] (IP)

where \(A \in \mathbb{Z}^{m \times n}\) and \(b \in \mathbb{Z}^m\).

We would like to know what are the properties that \(A\) and \(b\) must have such that the linear relaxation

\[
\begin{align*}
\text{minimize} & \quad c \cdot x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\] (LP)

always has an integral solution.

Recall that if the linear program has bounded objective, there must a basic feasible solution \(x\) that is optimal. Let \(B\) be a basis defining \(x\). Then \(x_B = A_B^{-1}b\), while all non-basic variables must be 0. It follows that if \(A_B^{-1}\) is an integral matrix then \(x\) must be integral as well. It turns out, as we shall prove shortly, that a sufficient condition for the integrality of \(A_B^{-1}\) is \(\det(A_B) \in \{1, -1\}\). This motivates the following definition.

**Definition 5.1.** We say a matrix \(A \in \mathbb{Z}^{m \times n}\) with full row rank is unimodular if the determinant of each basis of \(A\) is 1 or \(-1\).

First we will show that if the program (LP) is alway integral whenever \(A\) is unimodular, and that a weaker form of the converse also holds.

\(^1\) We have proved this for maximum weight bipartite matching, minimum vertex cover, and the minimum cut problems.
Theorem 5.1. A full row rank matrix \( A \in \mathbb{Z}^{m \times n} \) is unimodular if and only if the polyhedron \( \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \} \) is integral for all \( b \in \mathbb{Z}^m \).

Proof. We first prove the forward direction. Let \( x \) be an extreme point and \( B \) be a basis of \( x \). Our goal is to show that \( x \) is integral. From Cramer’s rule\(^1\) we know that \( A^{-1}_B = \frac{\text{Adj}(A_B)}{\det(A_B)} \), where \( \text{Adj}(A_B) \) is the adjugate\(^3\) of \( A_B \). Since \( A_B \) is integral, it follows that \( \text{Adj}(A_B) \) must be integral as well. Together with the fact that \( \det(A_B) \in \{1, -1\} \), this implies the integrality of \( A^{-1}_B \) and, therefore, that of \( x \).

The backward direction is surprisingly easy. Let \( B \) be a basis of \( A \). Our goal is to show that \( |\det(A_B)| = 1 \). Let \( b = A_B z + e_i \) where \( e_i \) is the unit vector with a single 1 in its \( i \)th coordinate and zeros elsewhere, \( i \) is an arbitrary index in \( \{1, \ldots, m\} \), and \( z \) is a suitable integral vector such that

\[
x_B = A^{-1}_B b = z + A^{-1}_B e_i \geq 0.
\]

This ensures that the basic solution induced by \( B \) is feasible. Since (LP) is integral it follows that the \( i \)th column of \( A^{-1}_B \) must be integral. Because \( i \) is arbitrary, it follows that every column of \( A^{-1}_B \) is integral. Therefore it must be that \( \det(A_B) \) and \( \det(A^{-1}_B) \) are integral. Together with the relation

\[
\det(A_B) \det(A^{-1}_B) = 1,
\]

we conclude that \( |\det(A_B)| = |\det(A^{-1}_B)| = 1 \).

This theorem can be generalized to linear programs that are not in standard form by refining the the property we require from the constraint matrix.

Definition 5.2. We say a matrix \( A \) is totally unimodular if every square submatrix of \( A \) has determinate 0, 1, or -1.

Theorem 5.2. A matrix \( A \in \mathbb{Z}^{m \times n} \) is totally unimodular if and only if the polyhedron \( \{ x \in \mathbb{R}^n : Ax \leq b, x \geq 0 \} \) is integral for all \( b \in \mathbb{Z}^m \).

Proof sketch. For simplicity, here we assume \( b \geq 0 \), but a similar argument holds for general \( b \in \mathbb{Z}^m \).

The forward direction is similar as Theorem 5.1. For the other direction, it suffices to argue that \( A \) is totally unimodular if and only if \( \begin{bmatrix} A & 1 \end{bmatrix} \) is unimodular, and that the extreme points of the polyhedron \( \{ x \in \mathbb{R}^n : Ax \leq b, x \geq 0 \} \) are integral if and only if the extreme points of the polyhedron \( \{ (x, z) \in \mathbb{R}^{n+1} : Ax + z = b, x \geq 0, z \geq 0 \} \) are integral.

Now that we have established the importance of these matrices, we shall explore some of their properties and alternative ways of proving that a given matrix is totally unimodular.

\(^1\) Cramer’s rule, Wikipedia.
\(^3\) The exact definition of the adjugate of a matrix is not important. The only thing you need to know is that the entries of the adjugate are defined in terms of determinants of submatrices of the original matrix.

Just for completeness, the adjugate of \( A_B \) is defined as the transpose of its cofactor matrix,

\[
\text{Adj}(A_B) = C^T,
\]

where \( c_{ij} \) is the \((i, j)\)-cofactor of \( A_B \), which is defined a \((-1)^{i+j} \) times the determinant of the submatrix of \( A_B \) that results from removing its \( i \)th row and its \( j \)th column.

Here are some examples to clarify our definitions so far.

The following matrix is unimodular

\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 1
\end{pmatrix},
\]

but the following matrix is not

\[
\begin{pmatrix}
1 & -1 & 1 \\
1 & 0 & -1
\end{pmatrix}.
\]

The following matrix is totally unimodular

\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
1 & 0 & -1
\end{pmatrix},
\]

but the following matrix is not

\[
\begin{pmatrix}
1 & -1 & 1 \\
1 & 0 & -1 \\
1 & 0 & 1
\end{pmatrix}
\]
since \( \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = -2 \).
5.2 Properties of Totally Unimodular Matrices

Let $A$ be a totally unimodular matrix. There are a few observations we can make about these matrices:

i) The entries of $A$ must be either 0, 1, or $-1$.

ii) The transpose of $A$ is also totally unimodular.

iii) The matrix $\begin{bmatrix} A \\ 1 \end{bmatrix}$ is also totally unimodular.

iv) The matrix $\begin{bmatrix} A \\ -A \end{bmatrix}$ is also totally unimodular.

A neat corollary\(^4\) of these properties is that if $A$ is totally unimodular then the polyhedron

$$\{ x \in \mathbb{R}^n_+ : a \leq Ax \leq b, 1 \leq x \leq u \}$$

is integral for all integral vector $a$, $b$, $l$, and $u$.

Another corollary is that if $c$ and $b$ are integral and $A$ is totally unimodular then both the primal program $\min \{ c \cdot x : Ax \geq b, x \geq 0 \}$ and the dual program $\max \{ b \cdot y : A^T y \leq c, y \geq 0 \}$ are integral.

5.3 Alternative Characterization

While Definition 5.2 is well suited to prove the integrality of the polyhedron associated with a constraint matrix $A$, it is not always straightforward to prove that a particular matrix $A$ is totally unimodular. In this section we introduce an alternative characterization of totally unimodular matrices that is easier to work with.

A bi-coloring of a matrix $M$ is a partition of its columns into red and blue columns. We say the bi-coloring equitable if for every row of $M$ the sum of its red entries differs from the sum of its blue entries by at most 1.

Theorem 5.3. A matrix $A$ is totally unimodular if and only every column-induced submatrix of $A$ admits an equitable bi-coloring.

The proof of this theorem is beyond the scope of this class. Here we will just make use of the theorem to prove that certain matrices are totally unimodular.

Lemma 5.1. The constraint matrix associated with the bipartite matching problem polyhedron

$$\{ x \in \mathbb{R}^{|E|}_+ : \sum_{e \in \delta(u)} x_e \leq 1 \ \forall u \in V \}$$

is totally unimodular.

Proof. Let $A$ be the constraint matrix associated with the polyhedron. Each row of $A$ is associated with a vertex $u$ in our bipartite

\(^4\)This corollary is a bit surprising. It means that if $P \subseteq \mathbb{R}^n_+$ is a polyhedron defined by a totally unimodular matrix, and $Q = [\ell_1, u_1] \times \cdots \times [\ell_n, u_n]$ then $P \cap Q$ is also an integral. In general, this is not the case.
If \( u \) is in the left shore of the graph, then we color the corresponding row red. If \( u \) is in the right shore of the graph, then we color the corresponding row blue.

Each column of \( A \) is associated with an edge \((u, v) \in E\), and it has a 1 in each of the rows associated with \( u \) and \( v \), and 0 elsewhere. It is easy to see that the bi-coloring is equitable for all row induced matrices of \( M \). Therefore the matrix is totally unimodular.

**Lemma 5.2.** The constraint matrix associated with the circulation problem polyhedron

\[
\left\{ f \in \mathbb{R}_+^{|E|} : \sum_{e \in \delta^{\text{in}}(u)} f_e = \sum_{e \in \delta^{\text{out}}(u)} f_e \quad \forall u \in V \right\}
\]

is totally unimodular.

**Proof.** Let \( A \) be the constraint matrix associated with the polyhedron. Each row of \( A \) is associated with a vertex \( u \in V \). We color red all rows. Each column of \( A \) is associated with an edge \((u, v) \in E\), and has a \(-1\) in the row associated with \( u \) and a 1 in the row associated with \( v \), and 0 elsewhere. It is easy to see that the bi-coloring is equitable for all row induced matrices of \( M \). Therefore the matrix is totally unimodular.

### 5.4 Subset Systems

We now turn our attention to a broad class of subset selection problems. These problems are defined by a pair \((U, I)\), where \( U \) is a universal set of elements and \( I \subseteq 2^U \) is a collection of feasible subsets of \( U \). The pair \((U, I)\) is called a *subset system* if for any \( S \subset T \subseteq U \) it holds that \( T \in I \) implies \( S \in I \).

Given a cost function \( c : U \to \mathbb{R}_+ \), the canonical optimization problem associated with \((U, I)\) is

\[
\text{maximize} \quad c(S) \\
\text{subject to} \quad S \in I
\]

Many natural problems fit under this framework. Some of them are hard and some, easy. We would like to identify the properties of \( I \) that make the corresponding optimization problem easy.

**Definition 5.3.** The rank function \( r : 2^U \to \mathbb{Z}_+ \) of the system \((U, I)\) is defined as follows

\[
r(S) = \max_{T \subseteq S : T \in I} |T|.
\]

It is easy to show that the following is an integer formulation for the canonical problem.

\[
\begin{align*}
\text{maximize} & \quad \sum_{j \in U} c_j x_j \\
\text{subject to} & \quad \sum_{j \in S} x_j \leq r(S) & \forall S \subseteq U \\
& \quad x_j \in \{0, 1\} & \forall j \in U
\end{align*}
\]

To get a formulation for the maximum s-t flow problem we just need to add the edge capacity constraints, which do not affect the total unimodularity of the constraint matrix, and maximize the linear objective \( f^t \).

Furthermore, since the transpose of the resulting matrix is also totally unimodular, it follows that the dual is also integral, which, as we saw last week, corresponds to the minimum s-t cut problem.

\footnote{In the optimization literature feasible sets are sometimes also called independent sets. Originally, a particular type of subset system was developed as an abstraction of linear independence:

\[
\begin{align*}
U & = \text{collection of vectors}, \\
I & = \left\{S \subseteq U : \text{vectors in } S \text{ are linearly independent} \right\}.
\end{align*}
\]

\footnote{For example, maximum weight spanning tree, maximum matching, and maximum weight independent set in graphs.}
As we shall prove shortly, the following property is key in arguing that the linear relaxation of (IP-IS) is integral.

**Definition 5.4.** The subset system \((U, \mathcal{I})\) is called a matroid if for any two subsets \(S, T \in \mathcal{I}\) such that \(|S| < |T|\) the following is true

\[
\exists x \in T \setminus S : S + x \in \mathcal{I}
\]  

Equation 5.1 is referred to as the matroid exchange axiom.

**Theorem 5.4.** Let \((U, \mathcal{I})\) be a matroid, then the following polyhedron is integral

\[
\left\{ x \in \mathbb{R}^{|U|} : x(S) \leq r(S) \forall S \subseteq U \right\}
\]

**Proof.** Let \(\hat{x}\) be a vertex of the polyhedron. Our goal is to show that \(x\) is integral. Let \(c\) be the cost vector that make \(\hat{x}\) the unique optimal solution of the linear program

\[
\begin{align*}
\text{maximize} & \quad \sum_{j \in U} c_j x_j \\
\text{subject to} & \quad \sum_{j \in S} x_j \leq r(S) \forall S \subseteq U \\
& \quad x_j \geq 0 \quad \forall j \in U
\end{align*}
\]

The dual program is given by

\[
\begin{align*}
\text{minimize} & \quad \sum_{S \subseteq U} r(S) y_S \\
\text{subject to} & \quad \sum_{S : j \in S} y_S \geq c_j \quad \forall j \in U \\
& \quad y_S \geq 0 \quad \forall S \subseteq U
\end{align*}
\]

Let us re-number the elements of our universal set \(U = \{1, \ldots, |U|\}\) so that \(c_j \geq c_{j+1}\) for \(j = 1, \ldots, |U| - 1\). We define a sequence of sets \(\emptyset = S^{(0)} \subseteq S^{(1)} \subseteq \cdots \subseteq S^{(|U|)} = U\) as follows

\[
S^{(j)} = \{1, \ldots, j\} \quad \text{for } j = 0, \ldots, |U|.
\]

Based on these sets we create a pair of primal and dual solutions

\[
x_j = r(S^{(j)}) - r(S^{(j-1)}) \quad \text{for } j = 1, \ldots, |U|,
\]

and

\[
y_S = \begin{cases} 
  c_j - c_{j+1} & S = S^{(j)} \text{ and } j = 0, \ldots, |U| - 1, \\
  c_{|U|} & S = S^{(|U|)}, \\
  0 & \text{otherwise.}
\end{cases}
\]

To show that \(\hat{x}\) is integral, we will argue that \(x = \hat{x}\). We will do this by proving the following properties:

i) \(\sum_{j \in U} c_j x_j = \sum_{S \subseteq U} r(S) y_S\)

ii) \(y\) is a feasible solution for the dual program

iii) \(x\) is a feasible solution for the primal program
These properties imply that $x$ is optimal, and since there is a unique optimal $x = \hat{x}$. It is clear that $x$ is integral, so the theorem follows.

Let us start with the first property

$$\sum_{j \in U} c_j x_j = \sum_{j=1}^{|U|} c_j \left( r(S^{(j)}) - r(S^{(j-1)}) \right)$$

$$= \sum_{j=1}^{|U|-1} (c_j - c_{j+1}) r(S^{(j)}) + c_{|U|} r(S^{(|U|)})$$

$$= \sum_{j=1}^{|U|} y_{S^{(j)}} r(S^{(j)})$$

$$= \sum_{S \subseteq U} y_S r(S)$$

For the second property we note that for all $j \in U$

$$\sum_{S : j \in S} y_S = \sum_{i=1}^{|U|} y_{S^{(i)}} = \sum_{i=1}^{|U|-1} (c_i - c_{i+1}) + c_{|U|} = c_j.$$

Notice that we have not used anything about the subset system to prove the first and the second properties.

For the last property we need to make use the matroid exchange axiom.

Exercises

1. Argue that the matrix $A$ is totally unimodular if and only if $\begin{bmatrix} A & 1 \end{bmatrix}$ is unimodular.

2. Let $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Argue that the polyhedron \( \{ x \in \mathbb{R}^n : Ax \leq b, x \geq 0 \} \) is integral if and only if the polyhedron \( \{ (x,z) \in \mathbb{R}^{n+m} : Ax + z = b, x \geq 0, z \geq 0 \} \) is integral.

3. Let $I_1, \ldots, I_n$ be a collection of open intervals of the real line. Each interval $I$ has associated a profit $p(I)$. The objective is to pick a maximum profit subset of intervals such that no two intervals overlap.

Consider the following program:

\[
\text{maximize} \quad \sum_{I} p(I) x_I \\
\text{subject to} \quad \sum_{I : p \in I} x_I \leq 1 \quad \forall p \in \mathbb{R} \\
x_I \in \{0,1\} \quad \forall I = I_1, \ldots, I_n
\]

This is not a proper integer linear program, as it contains an infinite number of constraint. Show that there is a set $Q$ of cardinality $2n - 1$ such that for any $q \in \mathbb{R}$ there is $p \in Q$ such that the constraints generated by $p$ and $q$ are equivalent.
Then show that the linear relaxation

\[
\begin{align*}
\text{maximize} & \quad \sum_{I} p(I) x_I \\
\text{subject to} & \quad \sum_{I: p \in I} x_I \leq 1 \quad \forall \ p \in Q \\
& \quad x_I \in [0, 1] \quad \forall \ I = I_1, \ldots, I_n
\end{align*}
\]

has a totally unimodular constraint matrix.

4. Prove that the subset system associated with the interval selection problem described in the previous exercise does not have the matroid property.

5. Let \( r \) be the rank function of a matroid \((U, I)\). Prove that for any \( S, T \subseteq U \) we must have

\[
r(S \cap T) + r(S \cup T) \leq r(S) + r(T).
\]

6. Let \((U, I)\) be the subset system defined by a set of vector \( U \) and the collection of linearly independent subsets of \( U \). Prove that the system is a matroid.

7. Let \((U, I)\) be a matroid and \( r \) be its rank function. Argue that the following linear program\footnote{Notice that for a matroid whose feasible sets correspond to acyclic subset of edges, the program corresponds to finding a minimum weight spanning tree.} is integral.

\[
\begin{align*}
\text{minimize} & \quad \sum_{j \in U} c_j x_j \\
x(S) & \leq r(S) \quad \forall S \subseteq U \\
x(U) & = r(U) \\
x_j & \geq 0
\end{align*}
\]
Solutions of selected exercises

1. Argue that the matrix $A$ is totally unimodular if and only if $[A I]$ is unimodular.

Solution. We show that for every square submatrix of $A$ there is a column-induced submatrix of $[A I]$ having the same determinant up to a sign change.

Let $M$ be the square submatrix of $A$ induced by the rows $R \subseteq \{1, \ldots, m\}$ and the columns $C \subseteq \{1, \ldots, n\}$. Let $T$ be the column-induced submatrix of $[A I]$ defined by the columns of $A$ indexed by $C$ and the columns of $I$ indexed by $\{1, \ldots, m\} \setminus R$. Notice that the relation is reversible—starting from $T$, we can construct $M$.

At this point, it should be easy to see that \( \det(M) = \pm \det(T) \). Indeed, we can compute $\det(T)$ by first expanding the determinant along the columns that come from $I$ until we are left with $M$.

It follows that the determinant of every square submatrix of $A$ is in \((-1,0,1)\) if and only if the determinant of every column-induced submatrix of $[A I]$ is in \((-1,0,1)\).

2. Let $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Argue that the polyhedron \( \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} \) is integral if and only if the polyhedron \( \{(x,z) \in \mathbb{R}^{n+m} : Ax + Iz = b, x \geq 0, z \geq 0\} \) is integral.

Solution. We establish a one-to-one integrality-preserving correspondence between the extreme points of the two polyhedra. Let us denote the two polyhedra with $P_1$ and $P_2$ respectively.

Let $x \in \mathbb{R}^n$ and $z = b - Ax$. Clearly, $x \in P_1$ if and only if $(x,z) \in P_2$. Furthermore, because $A$ and $b$ are integral, we know that $x$ is integral if and only if $(x,z)$ is integral. It only remains to show that $x$ is a basic feasible solution of $P_1$ if and only if $(x,z)$ is a feasible solution of $P_2$.

Suppose $x$ is a basic feasible solution of $P_1$. Consider a maximal set of linearly independent tight constraints at $x$. Let $R \subseteq \{1, \ldots, m\}$ be the indices of these constraints than come from the systems $Ax \leq b$ and $C \subseteq \{1, \ldots, n\}$ be the indices of these constraints that come from $x \geq 0$. Recall that because $x$ is basic, we know that it is the only solution that meets those constraints with equality.

Consider the basis $B$ formed by taking the columns of $I$ indexed by $R$ and the columns of $A$ indexed by $\{1, \ldots, n\} \setminus C$. It is not hard to see that $(x,z)$ is the basic feasible solution induced by the basis $B$.

Using a similar argument, we can justify the reverse direction: If $(x,z)$ is a basic feasible solution of $P_2$ then $x$ must be a basic feasible solution of $P_1$. 